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Elementary Particle States Based on the Clifford Algebra C_7

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The lepton isodoublet (e^-, ν_e), the "bare" nucleon isodoublet (n, p), and their antiparticles are shown to constitute a basis of the irreducible representation of the Clifford algebra C_7. The excited states of these doublets, i.e., (μ^-, ν_μ), $(\tau^-, \nu_\tau), \ldots$, and $(s^0, c^+), (b^0, t^+)$ are generated by the products $(e^-, \nu_e) \otimes a$ and $(n, p) \otimes a$, where $a \equiv 2^{-1/2}(e^- e^+ + \nu_e \bar{\nu}_e)$ has the same quantum numbers as the photon state. The bare baryons s, c, b, t carry the strangeness, charm, bottom, and top quantum numbers. These lepton and bare baryon states are in one-to-one correspondence with the integrally charged colored Han–Nambu quarks, and generate all the observed $su(3)$ and $su(4)$ hadron multiplets.

1. INTRODUCTION

Clifford algebras C_n (Clifford, 1878; Brauer and Weyl, 1935; Riesz, 1958; Hestenes, 1966; Kahan, 1966) play an important role in physics, as evidenced by the Pauli spin algebra C_2 and the Dirac algebra C_4. In increasing order, Eddington (1946) used C_4 in his fundamental theory, Barut and Haugen (1973) used C_6 in their formulation of conformally invariant massive spinor equations and the $e - \mu$ system, and Basri and Horwitz (1975) used C_7 to describe the hadronic mass spectrum. More recently, Casalbuoni and Gatto (1980) used higher-order Clifford algebras in a unified description of quarks and leptons. They use a gauge theory and orthogonal groups such as $o(13,1)$, that are generated by higher-order Clifford algebras.

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The main goal of this paper is to show how C_7 and its tensor products can be used to generate all observed particle multiplets; dynamics is outside the scope of this paper. In particular, we use C_7, its tensor products, and an orbital $o(4,2)$ algebra, to generate two sequences of isodoublets; one is the lepton sequence (e^-, ν_e), (μ^-, ν_μ), $(\tau^-, \nu_\tau), \ldots$, and the other is the baryon sequence (n, p), (s, c), $(b, t), \ldots$, where n, p are the "bare" nucleons, and s, c, b, t, are the "bare" hyperons carrying the "strangeness," "charm," "bottom," and "top" quantum numbers (QN), respectively. All particle states are obtained as tensor products of these states.

The essential properties of Clifford algebras and their physical identifications are outlined in Section 2. These principles are applied to C_7 in Section 3, and to the Dirac subalgebra in Section 4. The associated orbital algebra is presented in Section 5. Then the eigenstates of the complete algebra are given in Section 6. An irreducible representation (IR) of the isospin algebra, commuting with the Dirac algebra, is derived in Section 7. Excited lepton and bare baryon states are constructed in Section 8, meson states in Section 9, and baryon states in Section 10. It is shown in Section 10 that the lepton and "bare" baryon states play the role of the integrally charged colored Han–Nambu quarks. Regge trajectories are briefly discussed in Section 11, and the basic results are summarized in Section 12.

2. CLIFFORD ALGEBRAS

We outline here the basic facts about Clifford algebras necessary for this work. A (complex) Clifford algebra C_n is generated by the identity e, and n elements $e_1, \ldots, e_n$ satisfying the relations

$$e_A{}^2 = -e, \qquad e_A e_B = -e_B e_A \qquad \text{for } A \neq B \tag{1}$$

The remaining elements of C_n are obtained from all possible products of e_A. The number of elements that are a product of k different e_A is $\binom{n}{k} = n!/k!(n-k)!$, and the total number of elements of C_n is $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

For even n, there is no element besides the identity e that commutes with all the elements of C_n, i.e., the center of C_n consists of e only. However, for odd n, the center consists of e and the element $e_1 e_2 \cdots e_n$.

If we set

$$f_n \equiv i e_1 \cdots e_n \qquad \text{for } n = 1, 2, 5, 6 \tag{2a}$$

$$f_n \equiv e_1 \cdots e_n \qquad \text{for } n = 3, 4, 7 \tag{2b}$$

where $i = (-1)^{1/2}$, then it follows from (1) that

$$f_n^{\,2} = +e \tag{3}$$

and

$$P_n^{\pm} \equiv \tfrac{1}{2}(e \pm f_n) \tag{4}$$

are projectors (projection operators), i.e., they satisfy the relations

$$(P_n^{\pm})^2 = P_n^{\pm}, \qquad P_n^{+} P_n^{-} = P_n^{-} P_n^{+} = 0 \tag{5a}$$

$$P_n^{+} + P_n^{-} = e, \qquad P_n^{+} - P_n^{-} = f_n \tag{5b}$$

The first relation of (5a) shows that P_n are idempotent, and thus have the two eigenvalues, 0 and 1, only.

The set $C_n^{\,E}$ of all even elements (products of even numbers of e_A's) of C_n is a subalgebra of C_n isomorphic to C_{n-1}, i.e.,

$$C_n^{\,E} \sim C_{n-1} \subset C_n \tag{6}$$

but the set of odd elements of C_n is not a subalgebra. In the case of odd n, linear combinations of the even and odd elements can be constructed with the help of the projectors $P_n^{\pm}$ to form two disjoint subalgebras of C_n, namely,

$$C_{n-1}^{\;\pm} = P_n^{\pm} C_n^{\,E} \tag{7}$$

$$C_{n-1}^{\;+} C_{n-1}^{\;-} = 0, \qquad C_n = C_{n-1}^{\;+} \oplus C_{n-1}^{\;-} \qquad \text{for odd } n \tag{8}$$

This cannot be done for even n, since $P_n^{\pm}$ are not in the center.

The decomposition (8) plays a key role in our theory, since for each odd n it introduces an absolutely conserved, two-valued (± 1) QN that distinguishes between the two disjoint subspaces $C_{n-1}^{\;\pm}$. In particular, the decomposition of C_7 is identified with the lepton and baryon subspaces, that of C_5 with the charged and neutral subspaces representing spin-1/2 particle–antiparticle complexes.

3. THE INTRINSIC ALGEBRA C_7

The intrinsic algebra is defined to be the smallest Clifford algebra whose IR describes the minimum number of spin-1/2 particles that can be used to generate all other particles. C_4 yields the Dirac algebra D which describes a spin-1/2 particle–antiparticle complex. As stated at the end of the previous section, C_5 introduces the electric charge, C_6 completes the isospin algebra, and $C_7 = C_6^+ \oplus C_6^-$ distinguishes between leptons and baryons. We interpret the four Dirac complexes described by C_7 to be the lepton isodoublet (e^-, ν_e) and "bare" nucleon isodoublet (n, p).

The algebra C_7 is generated by the identity e and the seven elements $e_1, \ldots, e_7$ satisfying the relations (1). The total number of elements of C_7 is $2^7 = 2 \times 64 = 128$. According to (6), C_7 contains the following descending chain of even subalgebras:

$$\begin{aligned}
C_7{}^E &= \{e, e_{A_1}e_{A_2}, e_{A_1}\cdots e_{A_4}, e_{A_1}\cdots e_{A_6}\} = C_6, \qquad A_n = 1, \ldots, 7 \\
C_6{}^E &= \{e, e_{B_1}e_{B_2}, e_{B_1}\cdots e_{B_4}, e_1\cdots e_6\} = C_5, \qquad B_n = 1, \ldots, 6 \\
C_5{}^E &= \{e, e_{C_1}e_{C_2}, e_{C_1}\cdots e_{C_4}\} = C_4 \equiv D, \qquad C_n = 1, \ldots, 5 \\
C_4{}^E &= \{e, e_a e_b, e_1\cdots e_4\} = C_3, \qquad a, b = 1, \ldots, 4 \\
C_3{}^E &= \{e, e_j e_k\} = C_2, \qquad j, k = 1, 2, 3 \\
C_2{}^E &= \{e, e_1 e_2\} = C_1, \qquad C_1{}^E = \{e\} = C_0
\end{aligned} \tag{9}$$

The centers of C_7, C_5, C_3, and C_1 consist of e and, respectively, the elements

$$\begin{aligned}
e_1\cdots e_7 = f_7, \qquad i(e_1e_6)\cdots(e_5e_6) = ie_1\cdots e_6 = f_6 \\
(e_1e_4)\cdots(e_3e_4) = e_1\cdots e_4 = f_4, \qquad ie_1e_2 = f_2
\end{aligned} \tag{10}$$

It thus follows from (7) and (8) that

$$C_7 = C_6{}^+ \oplus C_6{}^-, \qquad C_6{}^\pm = P_7{}^\pm C_7{}^E \supset C_5{}^\pm, \qquad C_5{}^\pm = D^{\pm +} \oplus D^{\pm -} \tag{11a}$$

where

$$D^{\pm\pm} = P_7{}^\pm P_6{}^\pm C_5{}^E \tag{11b}$$

The further decomposition of each of the Dirac algebras D is discussed in the next section. We note here the existence of four such algebras describing ν_e, e^-, p, n and their antiparticles.

In accordance with the remarks made at the end of Section 2 and in this section, we have the following interpretation of eigenvalues:

$$f_7 = +1 \text{ for leptons, and } -1 \text{ for baryons}$$

$$f_6 = -1 \text{ for charged fermions, and } +1 \text{ for neutral ones} \tag{12}$$

The sign of f_7 is the same for fermions and antifermions, and $f_6 = -1$ for both negative and positive charge fermions. We call f_7 the *lepton–baryon* number, and f_6 the *charge number*. The lepton number L and baryon number B, as well as the electric charge Q are defined in Section 7.

4. THE INTRINSIC SPACE-TIME DIRAC ALGEBRA

The Dirac algebra $D \equiv C_5{}^E = C_4$ introduced in (9) is generated by the identity e and the four elements

$$d_a \equiv e_a e_5 \qquad (a = 1,2,3,4), \qquad d_0 \equiv i d_4 \tag{13}$$

In the IR of C_7, d_a are 16×16 matrices and are related to the usual 4×4 γ_μ matrices by the relations

$$d_\mu \equiv 1_4 \otimes \gamma_\mu, \qquad \mu = 0,1,2,3 \tag{14}$$

where

$$i\gamma_4 \equiv \gamma_0 = \gamma^0 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}, \qquad \gamma_j = -\gamma^j = \begin{pmatrix} 0_2 & -\sigma_j \\ \sigma_j & 0_2 \end{pmatrix} \tag{15}$$

and σ_j ($j = 1,2,3$) are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16}$$

The quantities 1_n and 0_n are the $n \times n$ unit and null matrices, respectively. The Dirac algebra in the context of C_4 has been investigated by many authors, and more recently by Greider (1980a, b), where other references can be found.

According to (1), the elements d_μ satisfy the relations

$$\{d_\mu, d_\nu\} \equiv d_\mu d_\nu + d_\nu d_\mu = 2 g_{\mu\nu} e \tag{17}$$

$$-g_{00} = g_{11} = g_{22} = g_{33} = -1, \qquad g_{\mu\nu} = 0 \qquad \text{for } \mu \neq \nu \tag{18}$$

The contravariant elements are defined by

$$d^{\mu}=g^{\mu\nu}d_{\nu}, \qquad g^{\mu\lambda}g_{\mu\nu}=\delta^{\lambda}{}_{\nu} \tag{19}$$

As usual, we define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \tag{20}$$

$$\gamma_5{}^2 = +1_4, \qquad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5 \tag{21}$$

Then

$$-1_4\otimes\gamma_5 = -id^0d^1d^2d^3 = id_0d_1d_2d_3 = e_1e_2e_3e_4 \equiv f_4 \tag{22}$$

$$f_4{}^2 = +e, \qquad f_4d_\mu = -d_\mu f_4 \tag{23}$$

The spin angular momentum (AM) tensor

$$S_{\mu\nu} \equiv \tfrac{1}{4}i\left[d_\mu, d_\nu\right] = \tfrac{1}{4}i[e_\mu, e_\nu] \equiv -S_{\nu\mu} \tag{24}$$

satisfies the commutation relations (CR)

$$\left[S_{\kappa\lambda}, d_\mu\right] = i(g_{\lambda\mu}d_\kappa - g_{\kappa\mu}d_\lambda) \tag{25}$$

$$\left[S_{k\lambda}, S_{\mu\nu}\right] = i(g_{\kappa\nu}S_{\lambda\mu} + g_{\lambda\mu}S_{\kappa\nu} - g_{\kappa\mu}S_{\lambda\nu} - g_{\lambda\nu}S_{\kappa\mu}) \tag{26}$$

The structure of D is clarified further with the help of the additional definitions

$$S_{\mu4} \equiv -S_{4\mu} \equiv \tfrac{1}{4}\left[d_\mu, f_4\right], \qquad S_{\mu5} \equiv -S_{5\mu} \equiv \tfrac{1}{2}d_\mu$$

$$S_{45} \equiv -S_{54} \equiv \frac{-i}{2}f_4 \tag{27}$$

The elements $S_{pq}(p,q=0,\ldots,5)$ satisfy the $o(4,2)$ CR's

$$\left[S_{pq}, S_{rs}\right] = i(g_{ps}S_{qr} + g_{qr}S_{ps} - g_{pr}S_{qs} - g_{qs}S_{pr}) \tag{28}$$

where

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = g_{55} = +1 \tag{29}$$

The intrinsic space-time Dirac algebra is thus given by

$$D \equiv C_5{}^E \equiv C_4 = \{e, d_\mu, S_{\mu\nu}, f_4 d_\mu, f_4\} = \{e, S_{pq}\} \cong o(4,2)_S \tag{30}$$

The four-dimensional nonunitary IR of $o(4,2)_S$ is shown in Section 6 to describe a spin-1/2 particle–antiparticle complex. This IR of $o(4,2)_S$ coincides with the *reducible* representation $D^{0\ 1/2} \oplus D^{1/2\ 0}$ of the Lorentz subalgebra (Barut, 1964).

Note that the electric charge Q does not enter the theory until the algebra is extended from C_4 to C_5. This is different from the interpretations by Greider (1980a, b) and some other authors, who introduce Q of the level of C_4. Support for our procedure is provided at the end of Section 7, where Q and other additive QN's are defined with the help of the particle–antiparticle projector obtained from C_4.

If we carry out the decomposition (8), we find that the subalgebra of the even elements of D,

$$C_4{}^E = \{e, S_{\mu\nu}, f_4\} = C_3 \tag{31}$$

is the Lorentz algebra extended by f_4. This algebra is generated by the "boosts" S_{0j} $(j = 1, 2, 3)$, and its even subalgebra is the Pauli spin algebra

$$C_3{}^E = \{e, S_{jk}\} = C_2 \tag{32}$$

$$2S_{jk} = 2S^{jk} = \epsilon^{jkl}(1_8 \otimes \sigma_l) \tag{33}$$

With the help of the *chirality* projector

$$P_c{}^{\pm} = P_4{}^{\pm} = \tfrac{1}{2}(e \pm f_4) \tag{34}$$

we may write

$$C_3 = C_2{}^- \oplus C_2{}^+, \qquad C_2{}^{\pm} = P_4{}^{\pm} C_2 \tag{35}$$

The Pauli algebra C_2 is in turn generated by S_{23} and S_{31}, and its even subalgebra is

$$C_1 \equiv C_2{}^E = \{e, S_{12}\} \tag{36}$$

The even subalgebra of C_1 is simply

$$C_0 \equiv C_1{}^E = \{e\} \tag{37}$$

By means of the spin projector

$$P_\sigma^{\pm} \equiv P_2^{\pm} = \tfrac{1}{2}(e \pm f_2), \qquad f_2 \equiv ie_1 e_2 = 2S_{12} \tag{38}$$

we may write

$$C_1 = C_0^- \oplus C_0^+, \qquad C_0^{\pm} \equiv P_\sigma^{\pm} C_0 \tag{39}$$

5. THE EXTRINSIC SPACE-TIME ALGEBRA

The Dirac algebra (30) is the *intrinsic* part of the space-time algebra. To describe the dynamics of elementary particles, we also need the extrinsic part of the space-time algebra, which is outside C_7. It is generated by the position and momentum 4-vectors

$$\mathbf{x} = (x^0, \ldots, x^3), \qquad x^0 \equiv ct, \qquad \mathbf{p} = (p^0, \ldots, p^3), \qquad p^0 \equiv E/c \tag{40}$$

As usual, the components of $\mathbf{x}$ and $\mathbf{p}$ satisfy the CR's ($\hbar = 1$)

$$[x^\mu, x^\nu] = 0, \qquad [p^\mu, p^\nu] = 0, \qquad [x^j, p_k] = i\delta^j{}_k \tag{41}$$

$$[x^\mu, \gamma_\nu] = 0, \qquad [p^\mu, \gamma_\nu] = 0 \tag{42}$$

In the equations of motion, p^0 is the Hamiltonian, and can be represented by $p^0 = p_0 = i\partial/\partial t \equiv i\partial_0$, when acting on a state $|\psi\rangle$. In the position representation, p^j can be represented (without restriction on the action domain) by $-p^j = p_j = i\partial/\partial x^j \equiv i\partial_j$. Thus, keeping these remarks in mind, one can write

$$p_\mu = i\partial/\partial x^\mu \equiv i\partial_\mu, \qquad p^\mu = g^{\mu\nu} p_\nu \tag{43}$$

The *orbital* AM is defined by

$$L_{\mu\nu} \equiv x_\mu p_\nu - x_\nu p_\mu \tag{44}$$

According to (41) and (42) it satisfies the same CR's (26) as $S_{\mu\nu}$, and in addition,

$$[L_{\kappa\lambda}, x_\mu] = i(g_{\lambda\mu} x_\kappa - g_{\kappa\mu} x_\lambda) \tag{45a}$$

$$[L_{\kappa\lambda}, p_\mu] = i(g_{\lambda\mu} p_\kappa - g_{\kappa\mu} p_\lambda) \tag{45b}$$

$$[L_{\kappa\lambda}, S_{\mu\nu}] = 0 \tag{46}$$

An *orbital* $o(4,2)_L$ algebra can be completed by defining further the components

$$L_{\mu 4} \equiv -L_{4\mu} \equiv x_\mu \mathbf{x}\cdot\mathbf{p} + \tfrac{1}{2}(1-\mathbf{x}^2)p_\mu$$

$$L_{\mu 5} \equiv -L_{5\mu} \equiv -x_\mu \mathbf{x}\cdot\mathbf{p} + \tfrac{1}{2}(1+\mathbf{x}^2)p_\mu$$

$$L_{45} \equiv -L_{54} \equiv \mathbf{x}\cdot\mathbf{p} = x^\mu p_\mu \tag{47}$$

where

$$\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x} = x^\mu x_\mu = (x^0)^2 - \vec{x}^2 \tag{48}$$

The components L_{pq} ($p,q=0,\ldots,5$) satisfy exactly the same CR's (28) as S_{pq}, and thus generate $o(4,2)_L$.

Corresponding to the two 4-vectors (40), there are the two Lorentz invariants:

$$\tilde{x} \equiv \mathbf{x}\cdot\mathbf{d} = x^\mu d_\mu, \qquad \tilde{p} \equiv \mathbf{p}\cdot\mathbf{d} = p^\mu d_\mu = p_\mu d^\mu \tag{49}$$

They satisfy the CR's

$$[S_{\mu\nu}, \tilde{x}] = -[L_{\mu\nu}, \tilde{x}] = i(x_\mu d_\nu - x_\nu d_\mu) \tag{50a}$$

$$[S_{\mu\nu}, \tilde{p}] = -[L_{\mu\nu}, \tilde{p}] = i(p_\mu d_\nu - p_\nu d_\mu) \tag{50b}$$

If we define the *total* AM by

$$J_{pq} \equiv S_{pq} + L_{pq} \tag{51}$$

we find

$$[J_{\mu\nu}, \tilde{x}] = 0, \qquad [J_{\mu\nu}, \tilde{p}] = 0 \tag{52}$$

which confirms the invariant character of (49). The components J_{pq} satisfy the same CR's (28) as S_{pq}, and thus generate the *total angular momentum* algebra $o(4,2)_J$. Moreover, $J_{\mu\nu}$ satisfy the same CR's (26) as $S_{\mu\nu}$, and

$$[J_{\kappa\lambda}, x_\mu] = i(g_{\lambda\mu}x_\kappa - g_{\kappa\mu}x_\lambda) \tag{53a}$$

$$[J_{\kappa\lambda}, p_\mu] = i(g_{\lambda\mu}p_\kappa - g_{\kappa\mu}p_\lambda) \tag{53b}$$

$$[J_{\kappa\lambda}, d_\mu] = i(g_{\lambda\mu}d_\kappa - g_{\kappa\mu}d_\lambda) \tag{53c}$$

From these relations it can be verified that

$$R \equiv \exp\left(\frac{i}{2}\theta J_{jk}\hat{n}^{jk}\right), \qquad B \equiv \exp\left(i\xi J_{0j}\hat{p}^{j}\right) \tag{54}$$

are, respectively, the operators for spatial rotation by an angle θ about an axis specified by the direction cosines $\hat{n}^{jk}$ (e.g., $\hat{n}^{12} = -\hat{n}^{21}$ is the projection of the rotation axis along the x^3 axis), and the Lorentz transformation (boost) along $\hat{p}^j$, where

$$\tanh\xi = \beta \equiv v/c = |\vec{p}|/E, \qquad \cosh\xi = (1-\beta^2)^{-1/2} \tag{55}$$

In general, for any quantity a, the Lorentz transformed quantity is

$$a' = LaL^{-1}, \qquad L \equiv \exp\left(\frac{i}{2}J_{\mu\nu}\theta^{\mu\nu}\right) \tag{56}$$

Note that $L_{\mu\nu}$ transform x^μ and p^μ, whereas $S_{\mu\nu}$ transform d_μ. If $a = \tilde{p} = p^\mu d_\mu$, then $\tilde{p}' = p'^\mu d'_\mu$ and both factors are transformed so that $\tilde{p}' = \tilde{p}$ is an invariant.

The intrinsic algebra C_7 is linked with the extrinsic space-time algebra through the two relations

$$\tilde{p} = p_\mu d^\mu, \qquad J_{\mu\nu} = S_{\mu\nu} + L_{\mu\nu} \tag{57}$$

where p_μ is given by (43), d^μ by (14), $S_{\mu\nu}$ by (24), and $L_{\mu\nu}$ by (44).

6. EIGENVALUES AND EIGENSTATES

From (40) and (17) we obtain

$$\tilde{p}^2 = p^\mu p^\nu d_\mu d_\nu = \tfrac{1}{2}\{d_\mu, d_\nu\}p^\mu p^\nu = g_{\mu\nu}p^\mu p^\nu = E^2 - \vec{p}^2 = m^2 = \mathbf{p}^2 \tag{58}$$

This means that for $m \neq 0$, $\tilde{p}$ has the two eigenvalues $\pm m$. If $\psi^\pm$ are the corresponding 16-component eigenspinors, then we have the generalized Dirac equation

$$\tilde{p}\psi^\pm = \pm m\psi^\pm, \qquad \tilde{p} \equiv p_\mu d^\mu \tag{59}$$

Note that $\psi^\pm$ are simultaneous eigenstates of $\tilde{p}^2$ and p_μ, i.e.,

$$\tilde{p}^2\psi^\pm = m^2\psi^\pm, \qquad p_\mu\psi_{p'}{}^\pm = p'_\mu\psi_{p'}{}^\pm, \qquad p_\mu = i\partial_\mu \tag{60}$$

We interpret ψ^{+} to be the state of a particle, and ψ^{-} the state of the corresponding antiparticle, both with $E>0$. This is different from the usual interpretation in which $\tilde{p}^2 = E^2 - \vec{p}^2 = m^2$ is solved for $E = \pm(p^2+m^2)^{1/2}$, and these roots are taken to be the eigenvalues of the Hamiltonian.

Note that (59) does not imply that the antiparticle has negative mass, since $\tilde{p}$ is not the mass operator. Instead, we are taking

$$\hat{p} \equiv \tilde{p}/m, \qquad \hat{p}\psi^{\pm} = \pm\psi^{\pm} \qquad \text{for } m \neq 0 \tag{61}$$

to be the particle–antiparticle operator; and thus

$$P^{\pm} \equiv \tfrac{1}{2}(e \pm \hat{p}) \tag{62}$$

are the Lorentz-invariant particle–antiparticle projectors. In the rest frame, $\hat{p} = d_0$, and its eigenvalue is the principal QN, n, introduced in Barut (1968a).

The Hermitian conjugate of (59) in the spinor space is

$$-\psi^{\dagger} d^{\mu\dagger} p_{\mu} = \pm m\psi^{\dagger}, \qquad p_{\mu}^{*} = -p_{\mu}$$

According to (14) and (15),

$$d^0 d^{\mu\dagger} d^0 = d^{\mu}, \qquad (d^{\mu})^2 = g_{\mu\mu} e \tag{63}$$

If we introduce the usual definition

$$\bar{\psi} \equiv \psi^{\dagger} d^0 \tag{64}$$

then we obtain

$$\bar{\psi}^{\pm}\tilde{p} = \mp\bar{\psi}^{\pm} m \tag{65}$$

This shows that the eigenstate of $\tilde{p}$ when it operates to the left is $\bar{\psi}$ (not $\psi^{\dagger}$) with eigenvalues opposite in sign to those when it operates to the right, as in (59).

To specify completely the eigenstates ψ, it is necessary to find a complete set of mutually commuting operators (CSCO) that include $\tilde{p}$ and p^{μ}. We first consider the case $m \neq 0$, where it is possible to set $p^{j} = 0$.

As usual, we introduce the spin 4-vector

$$w^{\kappa} \equiv \tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu} J_{\lambda\mu} p_{\nu} = \tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu} S_{\lambda\mu} p_{\nu}, \qquad \epsilon^{0123} = +1 \tag{66}$$

The second equality follows from (44) and (51), and $S_{\lambda\mu}$ is given by (24). If we define the 3-vectors

$$\vec{p}\equiv(p^1,p^2,p^3),\qquad \vec{S}\equiv(S_{23},S_{31},S_{12}),\qquad \vec{K}\equiv(S_{01},S_{02},S_{03}) \tag{67}$$

then it follows from (66) that

$$\mathbf{w}\equiv(w^0,w^1,w^2,w^3)=(-\vec{S}\cdot\vec{p},-p_0\vec{S}+\vec{p}\times\vec{K}) \tag{68}$$

In the rest frame where $\vec{p}=\vec{0}$, $p_0=E=m$,

$$\mathbf{w}=(0,-m\vec{S}),\qquad \vec{S}^2\psi=s(s+1)\psi\qquad \text{for } \vec{p}=\vec{0} \tag{69}$$

and

$$\mathbf{w}^2\equiv w^\kappa w_\kappa=-m^2s(s+1) \tag{70}$$

The last relation holds in any frame, since $\mathbf{w}^2$ is Lorentz invariant.

For $m\neq 0$, we adopt the following CSCO:

$$f_7,\, f_6,\, \hat{p}\equiv p^\mu d_\mu/m,\qquad \mathbf{p}^2=p^\mu p_\mu=m^2, \vec{\mathrm{p}}, \mathbf{w}^2$$

and

$$h\equiv -w^0/|\vec{p}|=\vec{S}\cdot\vec{p}/|\vec{p}|\quad \text{or}\quad S^3\equiv -w^3/m\qquad \text{for } \vec{p}=\vec{0} \tag{71}$$

where f_7, f_6 are given by (10) and (12), and h is the helicity. The operator f_7 distinguishes leptons from baryons, f_6 distinguishes charged from neutral fermions, and $\hat{p}$ distinguishes particles from antiparticles. The mass is given by $\mathbf{p}^2$, the linear momentum by $\vec{p}$, the spin by $\mathbf{w}^2$, and spin component along $\vec{p}$ by h or along x_3 by S^3.

Another possible CSCO for $m\neq 0$ is [see (34)–(39)]:

$$f_7,\, f_6,\, f_4,\, f_2, \mathbf{p}^2, \vec{p}, \mathbf{w}^2 \tag{72}$$

where f_4 replaces $\hat{p}$ and $f_2=2S^3$. The disadvantage of this set is that f_4 is not Lorentz invariant, and the eigenstates are superpositions of particle and antiparticle states.

For $m=0$, $\hat{p}\psi=0$, and $\hat{p}$ can no longer distinguish between particles and antiparticles. Moreover, we show below that the eigenvalues of f_4 and $-f_2$ are identical, and thus there are only two linearly independent eigenstates for $m=0$, as compared with four for $m\neq 0$. For this purpose, we choose the coordinate axes so that $\mathbf{p}=(p,0,0,p)$. Then [see (14) and (22)]

$$\tilde{p}\psi = p^\mu d_\mu \psi = p(d_0 + d_3)\psi = 0$$

implies

$$d_3\psi = -d_0\psi \quad \text{or} \quad d_3 d_0 \psi = d_0{}^2\psi = \psi \tag{73}$$

and

$$f_4\psi = +id_0 d_1 d_2 d_3 \psi = -id_1 d_2(d_3 d_0 \psi) = -f_2\psi \tag{74}$$

In this case f_4, $-f_2$, and $-2h$ have the same eigenvalues. We interpret f_4 as the particle–antiparticle operator for $m=0$. Which of the two eigenvalues ± 1 is to be associated with the particle is determined by experiment. The result is

$$f_4 = -1_4 \otimes \gamma_5 = \begin{cases} +1 & \text{for massless particles} \\ -1 & \text{for massless antiparticles} \end{cases} \tag{75}$$

Thus a massless particle always has negative helicity and its antiparticle positive helicity.

We take (72) to be the CSCO for $m=0$. Note that $\mathbf{p}^2 = \tilde{p}^2 = 0$ is diagonal, but $\tilde{p} = p^\mu d_\mu$ is not diagonal even though $\tilde{p}\psi = 0$. This is because $f_4 d_\mu = -d_\mu f_4$.

If ψ is a general state of C_7, the 16 eigenstates of C_7 may be projected out of it by means of the four projectors

$$P_7{}^{\pm} = \tfrac{1}{2}(e \pm f_7), \qquad P_6{}^{\pm} = \tfrac{1}{2}(e \pm f_6) \tag{76}$$

$$P^{\pm} = \tfrac{1}{2}(e \pm \hat{p}) \qquad \text{for } m \neq 0 \tag{77a}$$

$$P^{\pm} = P_4{}^{\pm} = \tfrac{1}{2}(e \pm f_4) \qquad \text{for } m = 0 \tag{77b}$$

$$P_h{}^{\pm} = \tfrac{1}{2}(e \pm 2h) \tag{78}$$

where $P^{\pm}$ is the particle–antiparticle projector, and $P_h{}^{\pm}$ is the helicity projector. Thus

$$\psi_A = P_A\psi, \qquad P_A = P_h{}^{\pm} P^{\pm} P_6{}^{\pm} P_7{}^{\pm}, \qquad A = 1,\ldots,16 \tag{79}$$

In particular,

$$P_7{}^+P_6{}^+\psi = |\nu_e\rangle, \qquad P_7{}^+P_6{}^-\psi = |e^-\rangle$$
$$P_7{}^-P_6{}^+\psi = |n\rangle, \qquad P_7{}^-P_6{}^-\psi = |p\rangle \tag{80}$$

The conservation of f_7, f_6, $\hat{p}$, and h can be deduced as follows: From (59), (60), and (65) we have

$$\begin{aligned} i\partial_\mu(\bar{\psi}d^\mu\psi) &= (\bar{\psi}\tilde{p})\psi + \bar{\psi}\,(\tilde{p}\psi) \\ &= (\mp m \pm m)\bar{\psi}\tilde{p}\psi = 0 \end{aligned}$$

This implies the continuity equation

$$i\partial_\mu(\bar{\psi}d^\mu\psi) = i\partial_0(\bar{\psi}d_0\psi) + i\partial_j(\bar{\psi}d^j\psi) = 0 \tag{81}$$

For either box normalization or a wave packet, ψ vanishes at the boundary of the normalization volume. By means of the divergence theorem, we obtain

$$\int \partial_j(\bar{\psi}d^j\psi)\,d_3x = \int(\bar{\psi}d^j\psi)\,dS_j = 0$$

where dS_j is an area element. Thus

$$\int \partial_0(\bar{\psi}d_0\psi)\,d_3x = \partial_0\int \psi^\dagger\psi\,d_3x = 0 \tag{82}$$

i.e., the normalization integral

$$\int \psi_A^\dagger\psi_A\,d_3x \qquad \text{is conserved} \tag{83}$$

where the states ψ_A in (83) stands for any of the states (79). Since

$$P_A{}^\dagger = P_A = P_A{}^2 \tag{84}$$

The statement (83) implies that

$$\langle P_A\rangle \equiv \int \psi^\dagger P_A\psi\,d_3x \qquad \text{is conserved} \tag{85}$$

TABLE I. Dichotomic Quantum Numbers of C_7

Name	Symbol	Value +1	Value −1
Lepton–baryon No.	$f_7 = e_1 \cdots e_7$	Leptons	Baryons
Charge No.	$f_6 = ie_1 \cdots e_6$	Charged	Neutral
Particle–antiparticle No.	$\hat{p} = p^{\mu} e_{\mu} e_5 / m$	Particle	Antiparticle
Helicity [see (71)]	$h = \vec{S} \cdot \vec{P} / \lvert \vec{P} \rvert$	$+1/2$	$-1/2$

Since $p_{\mu}, f_7, f_6, h, \hat{p}$ mutually commute, ψ, which is a superposition of momentum eigenstates, may be taken to be a simultaneous eigenstate of f_7, f_6, h, and $\hat{p}$. Thus (85) implies that the eigenvalues of f_7, f_6, h, and $\hat{p}$ are conserved. The physical interpretation of these QN's [see (14)] is summarized in Table I.

It is shown in the next section that L, B, Q, Y, and I_3 are functions of the operators f_7, f_6, $\hat{p}$ only, and consequently all their eigenvalues for free leptons and bare baryons are conserved.

7. THE INTERNAL ISOSPIN ALGEBRA

It was seen in (11a) that the intrinsic algebra C_7 is the direct sum of two C_6 algebras, one describing the leptons e^-, ν_e and their antiparticles, and the other the bare nucleons n, p and their antiparticles. We show now that each C_6 can be written as the direct product

$$C_6 = D \otimes C_2(I) \tag{86}$$

where $D = C_4 = C_5{}^E$ is the Dirac algebra discussed in Section 4, and $C_2(I)$ is a Clifford C_2 algebra that commutes with D.

We define

$$C_2(I) \equiv \{e, c_1, c_2\} \tag{87a}$$

where

$$c_1 \equiv ie_1 \cdots e_5 = i_2 \otimes \sigma_1 \otimes 1_4, \qquad c_2 \equiv ie_6 = 1_2 \otimes \sigma_2 \otimes 1_4 \tag{87b}$$

$$c_3 \equiv f_6 = ie_1 \cdots e_6 = 1_2 \otimes \sigma_3 \otimes 1_4 = -ic_1 c_2 \tag{87c}$$

Then

$$c_j^2 = +e, \qquad [c_j, c_k] = i\epsilon_{jk}{}^l c_l \tag{88}$$

$$c_j d_\mu = d_\mu c_j \tag{89}$$

Thus $C_2(I)$ is a Clifford algebra that commutes with D; and its direct product with D produces C_6. The full algebra C_7 is generated by C_6 and $P_7^\pm$.

The isospin components are identified with

$$I_j = \tfrac{1}{2} c_j \hat{p}, \qquad j = 1,2,3 \tag{90}$$

and they satisfy the CR's of the isospin $su(2)$ algebra

$$[I_j, I_k] = i\epsilon_{jk}{}^l I_l \tag{91}$$

If (87) is combined with (13), (14), and (22), i.e.,

$$d_a = e_a e_5 = 1_4 \otimes \gamma_a, \qquad f_4 = e_1 \cdots e_4 = -1_4 \otimes \gamma_5 \tag{92}$$

where $a = 1,2,3,4$, one obtains the complete IR of C_7,

$$\begin{aligned} e_a &= -d_a e_5 = i1_2 \otimes \sigma_i \otimes \gamma_5 \gamma_a, \qquad & e_5 &= f_4(-ic_1) = i1_2 \otimes \sigma_1 \otimes \gamma_5 \\ e_6 &= -ic_2 = -i1_2 \otimes \sigma_2 \otimes 1_4, \qquad & e_7 &= if_6 f_7 = i\sigma_3 \otimes \sigma_3 \otimes 1_4 \end{aligned} \tag{93}$$

From (93) and (10) we obtain

$$f_7 = \sigma_3 \otimes 1_8, \qquad f_6 = 1_2 \otimes \sigma_3 \otimes 1_4 \tag{94}$$

The particle projectors, according to (80), are

$$\begin{aligned} P(\nu_e) &= P_7^+ P_6^+ P^+, \qquad & P(e^-) &= P_7^+ P_6^- P^+ \\ P(n) &= P_7^- P_6^+ P^+, \qquad & P(p) &= P_7^- P_6^- P^+ \end{aligned} \tag{95}$$

The antiparticle projectors are obtained by replacing P^+ by P^-. By means of (95) we obtain for the lepton number L, baryon number B, electric charge Q, hypercharge Y, and I_3, the following expressions:

$$L = P(\nu_e) + P(e^-) - P(\bar{\nu}_e) - P(e^+) = P_7{}^+ \hat{p} \tag{96}$$

$$B = P(p) + P(n) - P(\bar{p}) - P(\bar{n}) = P_7{}^- \hat{p} \tag{97}$$

$$L + B = \hat{p}, \qquad L - B = f_7 \hat{p} \tag{98}$$

$$Q = -P(e^-) + P(p) + P(e^+) - P(\bar{p}) = -f_7 P_6{}^- \hat{p} \tag{99}$$

$$\begin{aligned} Y = &-P(\nu_e) - P(e^-) + P(p) + P(n) + P(\bar{\nu}_e) \\ &+ P(e^+) - P(\bar{p}) - P(\bar{n}) = -L + B = -f_7 \hat{p} \end{aligned} \tag{100}$$

$$\begin{aligned} 2I_3 = &\, P(\nu_e) - P(e^-) + P(p) - P(n) - P(\bar{\nu}_e) \\ &+ P(e^+) - P(\bar{p}) + P(\bar{n}) = f_6 f_7 \hat{p} \end{aligned} \tag{101}$$

The expression for Y is based on its interpretation as twice the average electric charge of an isomultiplet. This is reflected in the usual relation

$$Q = \tfrac{1}{2} Y + I_3 \tag{102}$$

which now follows automatically from the above definitions of Q, Y, and I_3.

Note that the opposite signs of the additively conserved QN's (96)–(101) for the particles and corresponding antiparticles originate completely from the projector $\hat{p}$; and this is a confirmation of our interpretation of $\hat{p}$ as the particle–antiparticle projector.

8. EXCITED STATES OF LEPTONS AND BARE BARYONS

We have seen that the basis of the IR of C_7 can be interpreted as consisting of the lepton isodoublet (e^-, ν_e), the bare baryons isodoublet (n, p), and their antiparticles.

We show in this and the following sections that the other particle states can be obtained by means of tensor products of C_7 by itself. Let us consider $C_7 \otimes C_7$, and in particular the states resulting from $(e^-, \nu_e) \otimes (e^+, \bar{\nu}_e)$. There are four such states: the isosinglet

$$a^0 \equiv 2^{-1/2}(e^- e^+ + \nu_e \bar{\nu}_e) \tag{103}$$

and the isotriplet

$$b^- \equiv e^- \bar{\nu}_e, \qquad b^0 \equiv 2^{-1/2}(e^- e^+ - \nu_e \bar{\nu}_e), \qquad b^+ \equiv \nu_e e^+ \tag{104}$$

The reason for the particular algebraic signs in a^0 and b^0 is as follows: Let $|I, I_3\rangle$ be an isospin eigenstate; and let $|1/2, \pm 1/2\rangle \equiv |\pm\rangle$. If C is the charge-conjugation operator, then

$$C|I, I_3\rangle = -(-1)^{I-I_3}|I, -I_3\rangle \tag{105}$$

where the minus sign in front is conventional. Consequently,

$$e^+ = Ce^- = C|-\rangle = |+\rangle, \qquad \bar{\nu}_e = C\nu_e = C|+\rangle = -|-\rangle \tag{106}$$

and the isospin states associated with a^0 and b^0 have the opposite signs from those in (103) and (104), as expected.

The spin of the states (103) and (104) is either zero or one. Their intrinsic parity is odd, since we have fermion–antifermion systems, and e^- and ν_e are assumed to have the same intrinsic parity. Thus the possible spectroscopic states are

$$^1S_0{}^-, \quad ^3S_1{}^-; \quad ^1P_1{}^+, \quad ^3P_{0,1,2}{}^+; \quad ^1D_2{}^-, \quad ^3D_{1,2,3}{}^-; \quad \cdots \tag{107}$$

where the "$\pm$" refers to the total parity.

In the $^3S_1{}^-$ and $^3D_1{}^-$ states, a^0 has the same QN's as the photon, and $b^{\pm,0}$ the same QN's as the vector mesons $W^{\pm,0}$. From now on, it will be understood that the states (103) and (104) have $J^P = 1^-$.

Let

$$f \equiv (e^-, \nu_e, n, p) \tag{108}$$

and consider the product fa. Since a is now bound with f, it need not have the same rest energy as in the free state. If the binding between f and a is magnetic, then the orbital AM, $L > 0$ (Barut, 1980a, 1982). For the smallest nonzero value $L = 1$, the total parity of fa is the same as f, since a has odd parity in the $J^P = 1^-$ state.

The algebra associated with fa (considered as a two-body system) consists of an external (center-of-mass) Poincaré algebra, and an internal $o(4,2)_J$ algebra (Barut, 1980b). For the most degenerate IR of the latter, the states are labeled by the total AM, J, and a "radial" principle QN, n_J (Barut and Reczka, 1977). The (J, n_J) spectrum is shown in Figure 1.

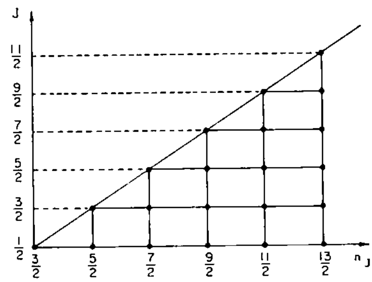


Fig. 1. The fermion most degenerate unitary IR of $o(4,2)$.

We restrict ourselves here to

$$L=1, \qquad \underline{J=1/2}, \qquad n_J=3/2, 5/2, 7/2, \quad \dots \tag{109}$$

The resulting states fa, which are considered to be excited states of f, are interpreted as follows:

$$\begin{aligned} &(e^- a)_{3/2}=\mu^-, &&(e^- a)_{5/2}=\tau^-, \quad \dots \\ &(\nu_e a)_{3/2}=\nu_\mu, &&(\nu_e a)_{5/2}=\nu_\tau, \quad \dots \\ &(na)_{3/2}=s^0, &&(na)_{5/2}=b^0, \quad \dots \\ &(pa)_{3/2}=c^+, &&(pa)_{5/2}=t^+, \quad \dots \end{aligned} \tag{110}$$

where the subscripts 3/2 and 5/2 are the n_J values, and s, c, b, t are the strange, charmed, bottom, and top bare baryons. Note that $n_J=3/2$ characterizes both (μ^-, ν_μ) and (s, c), while $n_J=5/2$ characterizes both (τ^-, ν_τ) and (b, t). Moreover, the parity of fa is the same as that of f.

By including the ground states (e^-, ν_e) and (n, p), we now have two sequences of isodoublets, the lepton sequence,

$$(e^-, \nu_e), \qquad (\mu^-, \nu_\mu), \qquad (\tau^-, \nu_\tau), \qquad \dots \tag{111}$$

and the bare baryon sequence,

$$(n, p), \qquad (s, c), \qquad (b, t), \qquad \dots \tag{112}$$

The term "exited states" of leptons is thus justified, as they are obtained from the ground state with one additional quantum of a photonlike state a.

By using a similar model to construct the lepton excited states, Barut (1980c) found it possible to derive a semiclassical formula for the lepton masses, namely,

$$\frac{M_N}{m_e} = 1 + \frac{3}{2}\alpha^{-1} \sum_{n=0}^{N} n^4 \tag{113}$$

where $N = 0, 1, 2 \cdots$ for $e, \mu, \tau, \ldots$.

Instead of taking the isodoublets in (111) as elements of an infinite-dimensional IR of $o(4,2) \otimes u(2)$, one can take $\{e^-, \nu_e, \mu^-, \nu_\mu\}$ as a basis of a $u(4)$ internal algebra, $\{e^-, \nu_e, \mu^-, \nu_\mu, \tau^-, \nu_\tau\}$ as a basis of a $u(6)$ internal algebra, etc. This is because the fundamental IR of $u(4)$ is four-dimensional, of $u(6)$ is six-dimensional, etc.; and $\{e^-, \nu_e\}$ is a basis of a $u(2)$ algebra, which is a subalgebra of $u(4)$, $u(6)$, etc.

Similarly, we take $\{n, p, s, c\}$ as a basis of a $u(4)$ algebra, $\{n, p, s, c, b, t\}$ as a basis of a $u(6)$ algebra, etc.

A fundamental IR of $u(4)$ decomposes into a $u(3)$-triplet and $u(3)$-singlet. If (e^-, ν_e) is retained as an isodoublet, then there are two ways of carrying out this decomposition, namely,

$$l \equiv (e^-, \nu_e, \mu^-; \nu_\mu) \text{ is the basis of the } (1)_4 \text{ IR of } u(4) \tag{114a}$$

$$l' \equiv (e^-, \nu_e, \nu_\mu; \mu^-) \text{ is the basis of the } (1^3)_{\bar{4}} \text{ IR of } u(4) \tag{114b}$$

where $(1)_4$ and $(1^3)_{\bar{4}}$ are the Young tableau designations of the IR's. In case l, (e^-, ν_e, μ^-) is a $u(3)$ triplet and ν_μ is a $u(3)$ singlet; whereas for l', (e^-, ν_e, ν_μ) is a $u(3)$ antitriplet and μ^- is a $u(3)$ antisinglet, as shown in Figure 2. We assume that each lepton state is an equal mixture of states

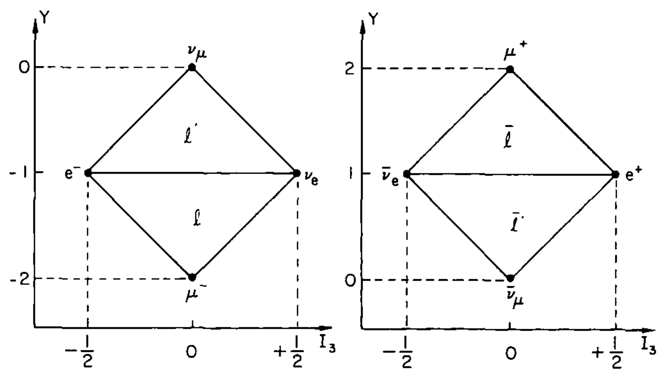


Fig. 2. The leptons and antileptons $u(4)$ quartets.

belonging to l and l'. In essence this assumes that leptons are not pure $u(4)$ and $u(3)$ eigenstates.

Bare baryons, on the other hand, are assumed to be pure $u(4)$ eigenstates (no b' is introduced), namely (see Figure 3),

$$b \equiv (n, p, s; c) \text{ is a basis of the } (1)_4 \text{ IR of } u(4) \tag{115}$$

This assumption, as well as that of mixing l and l', have to be justified dynamically.

In order to take spin into account, we take l, $\bar{l}'$, and b to be bases of the $(1)_8$ IR of

$$u(8) = u(4)_{\text{internal}} \otimes u(2)_{\text{spin}} \tag{116}$$

Then $\bar{l}$, l', and $\bar{b}$ are the bases of the $(1^7)_{\bar{8}}$ IR of $u(8)$.

If B is the baryon number, L the lepton number, Y the hypercharge, S the strangeness, C the charm, Q the electric charge, and I_3 the z component of isospin, then

$$Q = \tfrac{1}{2}Y + I_3, \qquad Y = B - L + C + S \tag{117}$$

The quantum numbers of l and l' are given in Table II.

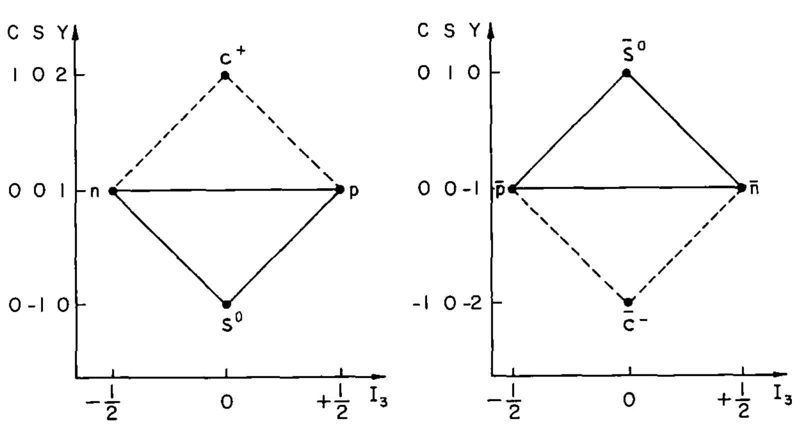


Fig. 3. The bare baryons and antibaryons $u(4)$ quartets.

TABLE II. Quantum Numbers of the Lepton $u(4)$ Quartet

	Q	Y	I_3	l: C	l: S	l': C'	l': S'
e^-	-1	-1	$1/2$	0	0	0	0
ν_e	0	-1	$-1/2$	0	0	0	0
μ^-	-1	-2	0	0	-1	-1	0
ν_μ	0	0	0	1	0	0	1

9. MESONS

In Section 8 we considered only four of the states resulting from $C_7 \otimes C_7$, namely, (103) and (104). Most of the states ff or $\bar{f}\bar{f}$ are not observed, and are presumably dynamically unstable, such as e^-e^- and pp. Few of these are stable, such as the duetron np. Since our main goal in this paper is to show how the observed families of states can be generated, we restrict our attention in this section to the observed meson states M, and in Section 10 to the baryon states B.

The meson states can be generated from $f\bar{f}$. For $b\bar{b}$, if the binding is the result of a strong interaction that allows $L=0$, then according to (107), the pseudoscalar (PS) mesons ($J^P=0^-$) are in the ${}^1S_0{}^-$ state, and the pseudovector (PV) mesons ($J^P=1^-$) are in the ${}^3S_1{}^-$ state. On the other hand, if the strong interaction is indeed magnetic, for which $L>0$, then a glance at (107) shows that the PV mesons must be in the ${}^3D_1{}^-$ state, while the PS mesons can be obtained from $f\bar{f}$ in combination with $\nu\bar{\nu}$, as shown in (128).

Accordingly, for magnetic binding, the most general expression for the PV mesons is

$$M_{\mathrm{PV}} = b\bar{b} \oplus l\bar{l} \oplus l'\bar{l}' \qquad \text{in } {}^3D_1{}^- \tag{118}$$

Since $b, l, \bar{l}'$ all belong to the same IR of $u(4)$, the resulting multiplets are exactly the same for all three terms. We take each meson state as the superposition of states having the same QN's, one from each of the terms in (118). The charm and strangeness of the states are derived uniquely from $b\bar{b}$.

The $l\bar{l}'$ and $l'\bar{l}$ products form the meson clouds of the baryons, and are discussed in Section 10.

According to (114), (115), and (116) each of the three terms in (118) yields the following IR's of $u(8)=u(4)\otimes u(2)$:

$$(1)_8 \otimes (1^7)_{\bar{8}} = (2, 1^6)_{63} \oplus (1^8)_{\bar{1}} \tag{119}$$

If the $u(8)$ IR's on the right-hand side of (119) are decomposed with respect to $u(4)\otimes u(2)$, one obtains (Itzykson and Nauenberg, 1966, Table C, pp. 118–119):

$$(1^8)_{\bar{1}} = (2^4)_{\bar{1}} \otimes (4^2)_{\bar{1}} = (2^4)_{\bar{1}} \qquad \text{of } u(4) \text{ with spin } 0 \tag{120}$$

$$\begin{aligned}(2,1^6)_{63} &= \left[(3,2^2,1)_{15}\otimes(4^2)_{\bar{1}}\right]\\ &\quad \oplus\left[(3,2^2,1)_{15}\otimes(5,3)_3\right]\oplus\left[(2^4)_{\bar{1}}\otimes(5,3)_3\right]\\ &= \left[(2,1^2)_{15}\,\text{spin}\,0\right]\oplus\left[(2,1^2)_{15}\,\text{spin}\,1\right]\oplus\left[(2^4)_{\bar{1}}\,\text{spin}\,1\right]\end{aligned} \tag{121}$$

The $u(3)$ content of the $u(4)$ IR $(2,1^2)_{15}$ is (Itzykson and Nauenberg, 1966, Table B, p. 112)

$$(2,1^2)_{15} = (2,1)_{8(0)}\oplus(1^3)_{\bar{1}(0)}\oplus(2,1^2)_{3(-1)}\oplus(1^3)_{\bar{3}(+1)} \tag{122}$$

where the subscripts $n(C)$ designate the dimension n of the $u(3)$ IR and its charm C.

The PV mesons resulting from (119) belong to the last two $u(4)$ multiplets of (121). Their content in terms of $b\bar{b}$, $l\bar{l}$, and $l'\bar{l}'$ is given below. The same expressions are applicable to the ${}^1S_0{}^-$ and ${}^3S_0{}^-$ states.

$(2,1)_{8(0)}$ of $u(3)$:

$$\rho^- = \left[n\bar{p}, e^-\bar{\nu}_e, e^-\bar{\nu}_e\right], \qquad \rho^+ = \left[p\bar{n}, \nu_e e^+, \nu_e e^+\right]$$

$$\rho^0 = 2^{-1/2}\left[n\bar{n} - p\bar{p}, e^-e^+ - \nu_e\bar{\nu}_e, e^-e^+ - \nu_e\bar{\nu}_e\right]$$

$$\begin{aligned}\omega_8 = 6^{-1/2}\big[&n\bar{n}+p\bar{p}-s\bar{s}, e^-e^+ + \nu_e\bar{\nu}_e - 2\mu^-\mu^+,\\ &e^-e^+ + \nu_e\bar{\nu}_e - 2\nu_\mu\bar{\nu}_\mu\big]\end{aligned} \tag{123}$$

$$K^{*0} = \left[n\bar{s}, e^-\mu^+, \nu_\mu\bar{\nu}_e\right], \qquad K^{*+} = \left[p\bar{s}, \nu_e\mu^+, \nu_\mu e^+\right]$$

$$\bar{K}^{*-} = \left[s\bar{p}, \mu^-\bar{\nu}_e, e^-\bar{\nu}_\mu\right], \qquad \bar{K}^{*0} = \left[s\bar{n}, \mu^-e^+, \nu_e\bar{\nu}_\mu\right]$$

$(1^3)_{\bar{1}(0)}$ of $u(3)$:

$$\omega_1 = 3^{-1/2}\left[n\bar{n}+p\bar{p}+s\bar{s}, e^-e^+ + \nu_e\bar{\nu}_e + \mu^-\mu^+, e^-e^+ + \nu_e\bar{\nu}_e + \nu_\mu\bar{\nu}_\mu\right] \tag{124}$$

The physical states $\omega(783)$ and $\phi(1020)$ are mixtures of ω_8 and ω_1, as usual.

$(1^3)_{\bar{3}(1)}$ of $u(3)$:

$$F^{*+}=\left[c\bar{s},\nu_\mu\mu^+,\mu^+\nu_\mu\right]$$

$$D^{*0}=\left[c\bar{p},\nu_\mu\bar{\nu}_e,\mu^+e^-\right],\qquad D^{*+}=\left[c\bar{n},\nu_\mu e^+,\mu^+\nu_e\right] \tag{125}$$

$(2,1^2)_{3(-1)}$ of $u(3)$:

$$F^{*-}=\left[\bar{c}s,\bar{\nu}_\mu\mu^-,\mu^-\bar{\nu}_\mu\right]$$

$$D^{*-}=\left[\bar{c}n,\bar{\nu}_\mu e^-,\mu^-\bar{\nu}_e\right],\qquad \bar{D}^{*0}=\left[\bar{c}p,\bar{\nu}_\mu\nu_e,\mu^-e^+\right] \tag{126}$$

$(2^4)_{\bar{1}(0)}$ of $u(4)$:

$$\omega_0=\tfrac{1}{2}\left[n\bar{n}+p\bar{p}+s\bar{s}+c\bar{c},e^-e^++\nu_e\bar{\nu}_e+\mu^-\mu^++\nu_\mu\bar{\nu}_\mu\right] \tag{127}$$

The physical state $\psi(3100)=[c\bar{c},\mu^-\mu^+,\nu_\mu\bar{\nu}_\mu]$ is a mixture of ρ^0,ω_8, ω_1,ω_0.

The $l\bar{l}$ and $l'\bar{l}'$ contents are identical for $\rho^\pm$, $F^{*\pm}$, and ω_0.

The $\Upsilon(9460)$ meson is obtained at the internal $u(6)$ level, and our analysis can be extended straightforwardly to this level.

The mixing of $l\bar{l}$ and $l'\bar{l}'$ in the meson states can be understood dynamically in terms of an exchange of a "virtual photon" (103), as in

$$K^{*+}=\nu_e\mu^+=\nu_e(e^+a)=(\nu_e a)e^+=\nu_\mu e^+$$

or in terms of a virtual annihilation–creation process $e^-e^+\rightleftarrows\nu_e\bar{\nu}_e$, as in

$$\begin{aligned}K^{*0}&=e^-\mu^+=e^-(e^+\nu_e\bar{\nu}_e)=(e^-e^+)\nu_e\bar{\nu}_e\\&=(\nu_e\bar{\nu}_e)\nu_e\bar{\nu}_e=(\nu_e\bar{\nu}_e\nu_e)\bar{\nu}_e=\nu_\mu\bar{\nu}_e\end{aligned}$$

The PS mesons can be generated from $b\bar{b}$ in the ${}^1S_0$ state for strong binding. For magnetic binding, however, $L>0$, and the simplest way they can be generated is by means of the formula

$$M_{\mathrm{PS}}=\left[(b\bar{b}\oplus l\bar{l}\oplus l'\bar{l}')\text{ in }{}^{1,3}P_{0,1,2}\right]\otimes\left[(\nu\bar{\nu})\text{ in }{}^3D_1{}^-\right] \tag{128}$$

with $L=0$ between the two factors in the tensor product. The second factor is the lightest component of the $u(4)$ singlet (127), and does not affect the $u(4)$ QN's of the first factor. Since the total parity of the first factor is even, that of the second is odd, and the orbital parity of the two factors is even, the total parity of M_{PS} is odd. The content of the first factor in terms of $f\bar{f}$ is exactly the same as in (123)–(127). The usual symbols for M_{PS} are

$$\pi^{-,0,+}, \eta, K^{0,+}, \overline{K}^{-,0}; \eta'; F^{+}, D^{0,+}; F^{-}, D^{-}, \overline{D}^{0}; \chi \tag{129}$$

These, as well as the corresponding M_{PV}, are shown in Figure 4. The additional $\nu\bar{\nu}$ pair in PS mesons (128) is analogous to the so-called $q\bar{q}$-sea terms in the quark model. This also helps to explain the decay channels of the mesons in a simple way—for example, the preference of the decay $\pi^- \to \mu^- \bar{\nu}_\mu$ over $\pi^- \to e^- \bar{\nu}_e$.

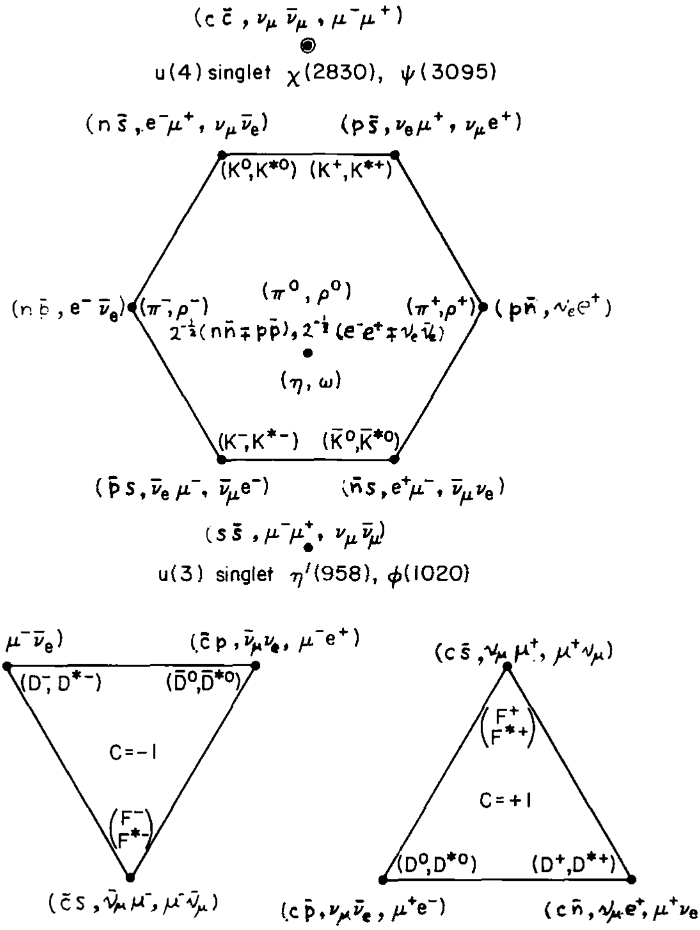


Fig. 4. The pseudoscalar and pseudovector mesons.

10. BARYONS

The observable baryon multiplets can be generated by

$$B = b \otimes l \otimes \bar{l}' \tag{130}$$

with the intuitive interpretation that b is the "bare" core defined by (115), and $M' = l\bar{l}'$ is the meson "cloud" around it.

Since b, l, and $\bar{l}'$ are distinct $u(4)$ quartets (even though l and l' consist of the same leptons), the baryon states are produced of three elements, one from each quartet. In this sense, our construction simulates the colored quark model construction

$$B = q(\text{green}) \otimes q(\text{blue}) \otimes q(\text{red}) \tag{131}$$

However, "color" is not an extra degree of freedom in our theory. The complete correspondence between b, l, $\bar{l}'$ and the quarks is given in Table III. The electric charges of the particles in Table III are in *exact* agreement with those of the integrally charged Han–Nambu quarks (Hendry and Lichtenberg, 1978, Table 2, p. 1713; cf. Han and Nambu, 1965). Moreover, the average charge $\langle Q \rangle$ of any three particles of the same flavor, is identical with the charge of the Gell-Mann–Zweig fractionally charged quark of the same flavor.

Since $\bar{l}'$ is a basis of the IR $(1)_8$ of (112), M' belongs to the following IR's of $u(8)$:

$$M' \equiv l \otimes \bar{l}' = (1)_8 \otimes (1)_8 = (2)_{36} \oplus (1^2)_{28} \tag{132}$$

Thus,

$$\begin{aligned} B &= \left[b(1)_8 \otimes M'(2)_{36}\right] \oplus \left[b(1)_8 \otimes M'(1^2)_{28}\right] \\ &= \left[(3)_{120} \oplus (2,1)_{168}\right] \oplus \left[(2,1)_{168} \oplus (1^3)_{56}\right] \end{aligned} \tag{133}$$

TABLE III. Correspondence between b, l, $\bar{l}'$, and Quarks

	b	l	$\bar{l}'$	
u	p	ν_e	e^+	$2/3$
d	n	e^-	$\bar{\nu}_e$	$-1/3$
s	s^0	μ^-	$\bar{\nu}_\mu$	$-1/3$
c	c^+	ν_μ	μ^+	$2/3$
Flavor / Color	Green	Blue	Red	$\langle Q \rangle$

The $u(4)_{\text{internal}} \otimes u(2)_{\text{spin}}$ decompositions of the resulting IR's are (Itzykson and Nauenberg, 1966, Table C, p. 115):

$$(3)_{120} = (2,1)_{20}\,\text{spin}\tfrac{1}{2} \oplus (3)_{20}\,\text{spin}\tfrac{3}{2} \tag{134a}$$

$$\begin{aligned}(2,1)_{168} = {}&(1^3)_{\bar{4}}\,\text{spin}\tfrac{1}{2} \oplus (3)_{20}\,\text{spin}\tfrac{1}{2}\\ &\oplus (2,1)_{20}\,\text{spin}\tfrac{1}{2} \oplus (2,1)_{20}\,\text{spin}\tfrac{3}{2}\end{aligned} \tag{134b}$$

$$(1^3)_{56} = (1^3)_{\bar{4}}\,\text{spin}\tfrac{3}{2} \oplus (2,1)_{20}\,\text{spin}\tfrac{1}{2} \tag{134c}$$

Finally, the $u(3)$ decompositions of the $u(4)$ IR's on the right-hand side of (134) are (Itzykson and Nauenberg, 1966, Table B, p. 112):

$$\begin{aligned}(2,1)_{20} &= (2,1)_{8(0)} \oplus (2)_{6(1)} \oplus (1^2)_{\bar{3}(1)} \oplus (1)_{3(2)}\\ (3)_{20} &= (3)_{10(0)} \oplus (2)_{6(1)} \oplus (1)_{3(2)} \oplus (\cdot)_{1(3)}\\ (1^3)_{\bar{4}} &= (1^3)_{\bar{1}(0)} \oplus (1^2)_{\bar{3}(1)}\end{aligned} \tag{135}$$

The subscripts $n(C)$ denote the dimension n and charm C of the $u(3)$ IR's.

In order to figure out the precise content of (135) we note that the $u(4) \otimes u(2)$ decompositions of M' are

$$\begin{aligned}M'(2)_{36} &= (1^2)_6\,\text{spin}\,0 \oplus (2)_{10}\,\text{spin}\,1\\ M'(1^2)_{28} &= (2)_{10}\,\text{spin}\,0 \oplus (1^2)_6\,\text{spin}\,1\end{aligned} \tag{136}$$

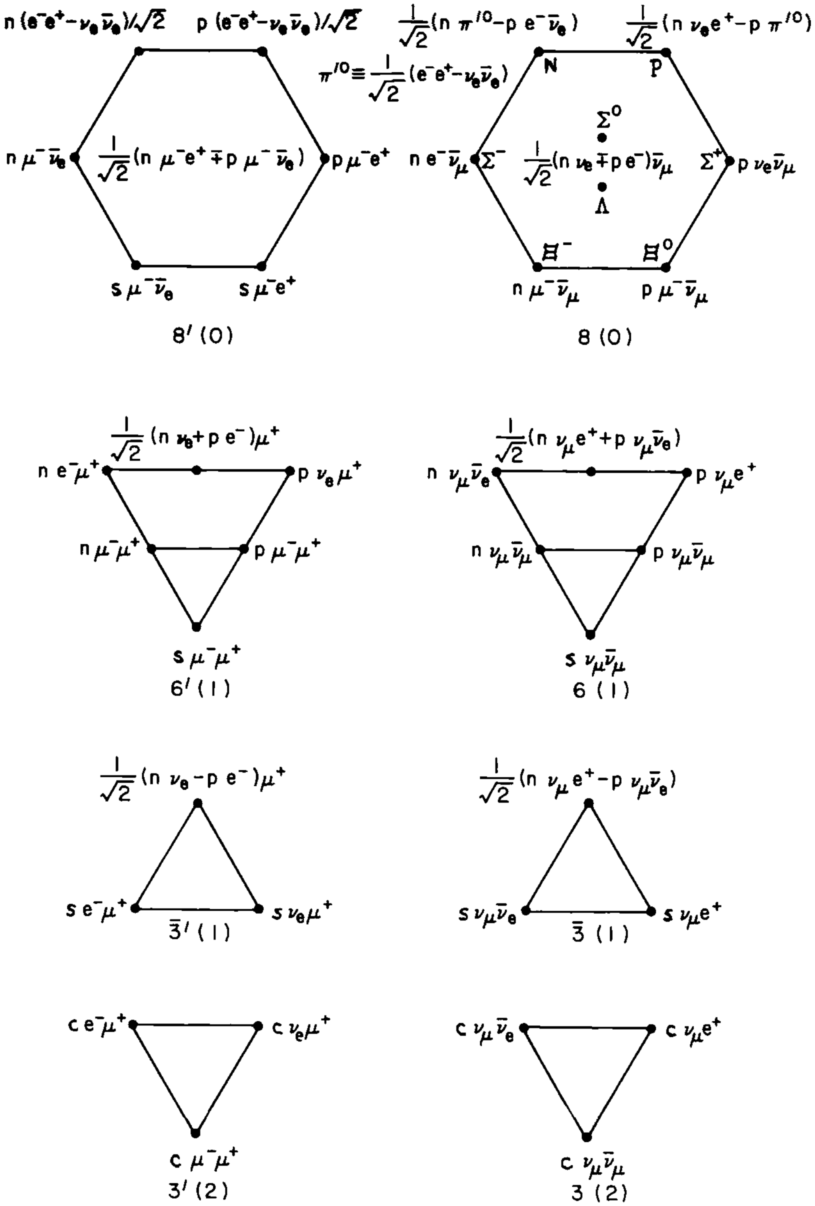
Moreover, the $u(3)$ decompositions of the $u(4)$ multiplets on the right-hand side of (136) are

$$(1^2)_6 = (1^2)_{\bar{3}} \oplus (1)_3, \qquad (2)_{10} = (2)_6 \oplus (1)_3 \oplus (\cdot)_1 \tag{137}$$

From (137), the product $l(\nu_e, e^-, \mu^-; \nu_\mu)_4 \otimes \bar{l}'(e^+, \bar{\nu}_e, \bar{\nu}_\mu; \mu^+)_4$ and the QN's listed in Table II, we obtain

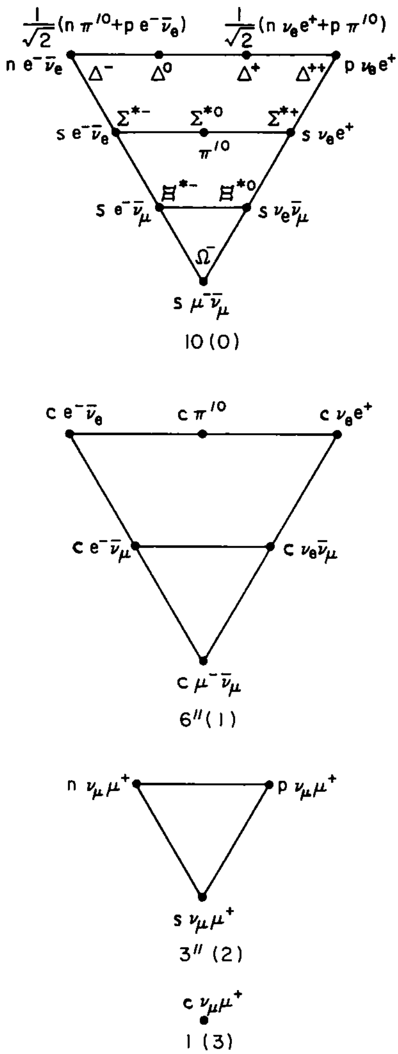
$$\begin{aligned}(1^2)_6 = {}&M'\big(2^{-1/2}(e^-e^+ - \nu_e\bar{\nu}_e); \mu^-e^+, \mu^-\bar{\nu}_e\big)_{\bar{3}(0)}\\ &\oplus M'(\nu_e\mu^+, e^-\mu^+; \mu^-\mu^+)_{3(1)}\end{aligned} \tag{138a}$$

$$\begin{aligned}(2)_{10} = {}&M'\big(\nu_e e^+, \pi'^0 \equiv 2^{-1/2}(e^-e^+ - \nu_e\bar{\nu}_e), e^-\bar{\nu}_e; \nu_e\bar{\nu}_\mu, e^-\bar{\nu}_\mu; \mu^-\bar{\nu}_\mu\big)_{6(0)}\\ &\oplus M'\big(\nu_\mu e^+, \nu_\mu\bar{\nu}_e; \nu_\mu\bar{\nu}_\mu\big)_{3(1)} \oplus M'\big(\nu_\mu\mu^+\big)_{1(2)}\end{aligned} \tag{138b}$$

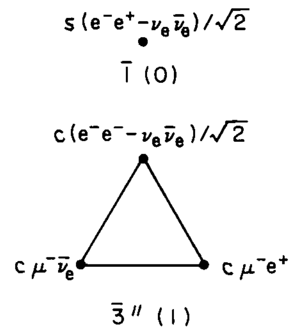


(a)

Fig. 5. (a) The baryon $u(4)$ multiplets $(2,1)'_{20}$ and $(2,1)_{20}$.



(b)



(c)

Fig. 5. (b) The baryon $u(4)$ multiplet $(3)_{20}$. (c) The baryon $u(4)$ multiplet $(1^3)_{\bar{4}}$.

Each term in (138) is a $u(3)$ multiplet $n(C)$. Within each term, the isospin multiplets are separated by semicolons, and ordered in decreasing value of Y; and for each isomultiplet, the M' states are listed in decreasing value of I_3. It is important to note that the leptons in (138) have the QN's of l and the antileptons of $\bar{l}'$. Thus although the constituents are the same as in Figure 4, the states are different, as indicated by the label M'.

Finally, multiplying $b(p, n; s)_{3(0)}$ and $b(c)_{1(1)}$ by (138), we obtain the baryon multiplets shown in Figure 5, and classified as indicated by (135). These sets of multiplets occur several times with spins 1/2 and 3/2 as given in (134).

In Figure 5a, the $u(4)$ multiplet $(2,1)_{20}$ occurs twice. In the occurrence on the right-hand side, the total mass of the constituents of each state is less than that of the corresponding state on the left-hand side occurrence. For this reason, we identify the well-known baryon spin-1/2 octet $B(P, N; \Sigma^{\pm,0}; \Lambda; \Xi^{0,-})$ with the 8(0) $u(3)$-multiplet of the $(2,1)_{20}$ IR of $u(4)$. The baryon spin-3/2 decouplet $B(\Delta, \Sigma^*, \Xi^*, \Omega)$ is identified with the 10(0) $u(3)$-multiplet of the $(3)_{20}$ IR of $u(4)$. Both of these multiplets belong to the $(3)_{56}$ IR of zero-charm baryons of $u(6)$, which in turn belongs to the $(3)_{120}$ IR of $u(8)$ given in (134a).

As in Section 8, the combination of the baryon core b with its cloud M' leads to an $o(4,2)$ algebra. It has been shown (Barut, 1968b–d) that this algebra can produce the correct baryon mass spectrum.

11. HIGHER SPIN STATES AND REGEE TRAJECTORIES

The hadron states constructed in Sections 9 and 10 were assumed to be in the lowest possible energy state. By considering orbital excitations, one can generate a sequence of states with the same $u(4)$ QN's and higher values of J.

The masses of these states can be calculated from, e.g., infinite component wave equations, whose parameters contain the dynamical information about the interaction of the constituents. Linear Regge trajectories can be obtained in this way, as shown in Barut and Reczka (1977, p. 613).

Another approach to the higher angular momentum states, is the dynamical approach, such as that used in Barut (1980c). For example, the magnetic forces between the leptons e^-, $\bar{\nu}_e$ can be used to obtain ρ, A_2, etc.

The fact that these two approaches, one dynamical and the other via a wave equation, are possible and identical, has been demonstrated in the case of known composite systems such as the hydrogen atoms. These problems will be discussed separately.

12. CONCLUDING REMARKS

There are several important results that emerge from the present work:

(a) The four basic pure Dirac fermion states e^-, ν_e, n, p form a basis of the IR of the Clifford algebra C_7.

(b) All observed particle states can be generated from tensor products of C_7 by itself, along with an orbital $o(4,2)$ algebra. Thus in effect, all particles are constructed from the four particles e^-, ν_e, n, p.

The introduction of the bare neutron n as a building block alongside the three particles e^-, ν_e, p, is a consequence of the use of C_7. There are models (Barut, 1980c) in which n is first constructed from $pe^-\bar{\nu}_e$, and then used as a building block.

(c) The isodoublets $(\mu^-, \nu_\mu), (\tau^-, \nu_\tau), \ldots$, are shown to be excited states of (e^-, ν_e); and $(s, c), (b, t), \ldots$, are shown to be excited states of (n, p).

(d) The elements of the sequence of lepton isodoublets (e^-, ν_e), $(\mu^-, \nu_\mu), (\tau^-, \nu_\tau), \ldots$ are distinguished by the same QN that distinguishes the elements of the sequence of bare baryons $(n, p), (s, c), (b, t), \ldots$.

(e) The $u(4)$ quartets $(n, p, s; c), (e^-, \nu_e, \mu^-; \nu_\mu), (e^+, \bar{\nu}_e, \bar{\nu}_\mu; \mu^+)$ are in one-to-one correspondence with the integrally charged colored Han–Nambu quarks in constructing the hadrons.

(f) All particle multiplets derived in this work are the same as those obtained in the quark model.

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